

An Expandable Local and Parallel Two-Grid Finite Element Scheme

Yanren Hou*, GuangZhi Du†

Abstract

An expandable local and parallel two-grid finite element scheme based on superposition principle for elliptic problems is proposed and analyzed in this paper by taking example of Poisson equation. Compared with the usual local and parallel finite element schemes, the scheme proposed in this paper can be easily implemented in a large parallel computer system that has a lot of CPUs. Convergence results base on H^1 and L^2 a priori error estimation of the scheme are obtained, which show that the scheme can reach the optimal convergence orders within $|\ln H|^2$ or $|\ln H|$ two-grid iterations if the coarse mesh size H and the fine mesh size h are properly configured in 2-D or 3-D case, respectively. Some numerical results are presented at the end of the paper to support our analysis.

Key Words two-grid finite element method, domain decomposition method, local and parallel algorithm, error estimation
MSC2000 65N15, 65N30, 65N55

1 Introduction

Two-grid or multi-grid finite element methods and domain decomposition methods are powerful tools for numerical simulation of solutions to PDEs with high resolution, which are otherwise inaccessible due to the limits in computational resources. For examples, the domain decomposition schemes, nonlinear Galerkin schemes and two-grid/two-level post-processing schemes in [2, 4, 5, 7, 9, 12, 13, 16, 17, 20] and the references therein. In the past decade, a local and parallel two-grid finite element method for elliptic boundary value problems was initially proposed in [21] and was extended to nonlinear elliptic boundary value problems in [22] and Stokes and Navier-Stokes equations in [10, 11].

Let us briefly recall the local and parallel two-grid finite element method in [21] for the following simple Poisson equation with Dirichlet boundary condition

*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China. (yrhou@mail.xjtu.edu.cn)

†School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China.

defined in convex domain $\Omega \subset R^d$, $d = 2, 3$:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

whose weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \quad (1.2)$$

Suppose $T^H(\Omega)$ is a regular coarse mesh triangulation of Ω and $S^H(\Omega) \subset H^1(\Omega)$, $S_0^H(\Omega) = S^H(\Omega) \cap H_0^1(\Omega)$ are the corresponding finite element spaces defined on $T^H(\Omega)$. Let us decompose the entire domain Ω into a series of disjoint subdomains, $\bar{\Omega} = \bigcup_{j=1}^N \bar{D}_j$. For example, see Fig. 1.

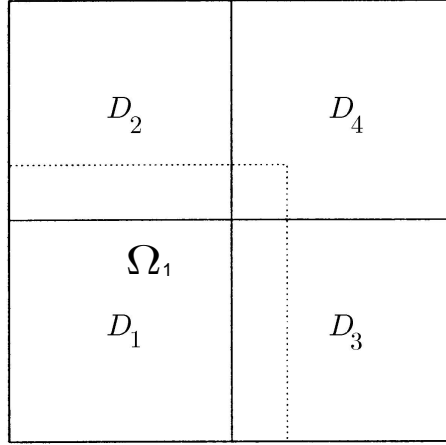


Fig. 1: Decomposition of the domain Ω

If the coarse mesh standard Galerkin approximation $u_H \in S_0^H(\Omega)$ is obtained, by expanding each subdomain D_j to another subdomain $\Omega_j \subset \Omega$ and for a given fine mesh size $h < H$, one of the local and parallel two-grid schemes proposed in [21] is: find $e_h^j \in S_0^h(\Omega_j)$ such that

$$a(e_h^j, v) = (f, v) - a(u_H, v), \quad \forall v \in S_0^h(\Omega_j). \quad (1.3)$$

And the final approximation u^h is defined piecewisely by

$$u^h = u_H + e_h^j, \quad \text{in } D_j, \quad j = 1, 2, \dots, N.$$

Error estimations in [21] shows that u^h can reach the optimal convergence order in H^1 norm. However, it is obvious that u^h is in general discontinuous

and its L^2 error bound does not in general have higher order than its H^1 error bound. To overcome this defect of the algorithm, the authors in [21] modified the above scheme to ensure the continuity of u^h in Ω and finally do a coarse grid correction to get the optimal error bound in L^2 norm. The most attractive feature of the algorithm is that the series of subproblems are independent once u_H is known and therefore it is a highly parallelized algorithm.

On the other hand, one can easily see from the content of [21] that their error estimates heavily depend on the usage of the superapproximation property of finite element spaces. Thanks to [18], we know that the usage of this property makes the error constant appeared in [21] has the form $O(t^{-1})$, where $t = \text{dist}(\partial D_j \setminus \partial\Omega, \partial\Omega_j \setminus \partial\Omega)$. To guarantee the error orders obtained in [21], one should demand that $t = O(1)$. That means the distance between the boundaries of a specific subdomain D_j and its expansion Ω_j should be of constant order. Therefore Ω_j could not be arbitrary small even when $\text{diam}(D_j)$ tends to zero. This will lead to a vast waste of parallel computing resources.

In this paper, we follow the basic idea presented in [21] to construct another form of two-grid local and parallel scheme, in which the scale of each subproblem can be much smaller compared with that in [21]. In fact, we will deal with the case of $\text{diam}(D_j) = O(H)$ and $t = O(H)$. We call the scheme an expandable local and parallel two-grid scheme because the scale of each subproblem can be arbitrary small as H tends to zero and every two adjacent subproblems only have a small overlapping. Similarly, to get a better L^2 error bound, a coarse grid correction is done in each cycle of two-grid iteration.

Different from the previously mentioned local and parallel schemes, we use superposition principle to generate a series of local and independent subproblems and this will make the global approximation continuous in Ω . Such kind of technique has been successfully used in [14, 15], in which adaptive variational multi-scale methods were constructed. In fact, the scheme in this paper is quite similar to the variational multi-scale schemes in the two references. The difference is the schemes presented in [14, 15] are adaptive schemes based upon some a posteriori error estimates and therefore some boundaries related problems have to be solved. Another contribution of this paper compared with [14, 15] is that a priori error estimate of the scheme is obtained and for patches of given size, our analysis show that a few iterations, say $O(|\ln H|^2)$ or $O(|\ln H|)$ in 2-D or 3-D respectively, will generate an approximation with same accuracy as the fine mesh standard Galerkin approximation. In addition, following the idea of partition of unity method (see [3]), authors in [19, 23] proposed a local and parallel two-grid scheme for second order linear elliptic equations in 2-D case based on the scheme presented in [21]. Although the usage of partition of unity method makes the global approximation continuous, but their error estimation is still based on the superapproximation property of the finite element space and therefore the distance t , theoretically, must be constant order to guarantee their estimations.

The rest of this paper is organized as follows. In the coming section, some preliminary materials are provided. In section 3, local and parallel scheme is constructed. Error estimates in both H^1 and L^2 norms are obtained for the

scheme in section 4. Finally, some numerical experiments are given to support our analysis in section 5.

2 Preliminaries

In this paper, for the sake of simplicity of analysis, we only consider the case of the Poisson equation (1.1) and we can get similar results for general linear elliptic problems as the Poisson equation with little modifications.

For a bounded convex domain $\Omega \subset R^d$, $d = 2, 3$, we use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms, see, e.g., [1] and [6]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ and $|\cdot|_{s,\Omega}$ the corresponding semi-norm. In some places of this paper, $\|\cdot\|_{s,\Omega}$ should be viewed as piecewisely defined if it is necessary. For simplicity, the following symbols \lesssim , \gtrsim and \approx will be used in this paper. In the rest, $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1 , c_2 , c_3 and C_3 that are independent of mesh size, x_i , y_i and local domains which will be introduced in the following sections. In the following, we denote by (\cdot, \cdot) the L^2 -inner product on Ω . Thus, $\|\cdot\|_{0,\Omega} = (\cdot, \cdot)^{\frac{1}{2}}$ and, in $H_0^1(\Omega)$, we know that $\|\cdot\|_{1,\Omega} \approx \|\nabla \cdot\|_{0,\Omega}$. For simplicity of expression, we use $\|\cdot\|_{\Omega}$ to denote $\|\cdot\|_{0,\Omega}$ in the rest. For subdomains $S_1 \subset S_2 \subset \Omega$, $S_1 \subset\subset S_2$ means that $\text{dist}(\partial S_2 \setminus \partial\Omega, \partial S_1 \setminus \partial\Omega) > 0$.

For any given $S_1 \subset \Omega$, we denote by $(\cdot, \cdot)_{S_1}$ the L^2 -inner product on S_1

$$a(u, v)_{S_1} = (\nabla u, \nabla v)_{S_1}, \quad a(u, v) = (\nabla u, \nabla v)_{\Omega},$$

in the rest of this paper. Then we get the weak form (1.2).

It is obvious that

$$\|\nabla u\|_{S_1}^2 = a(u, u)_{S_1} \leq \|\nabla u\|_{S_1} \|\nabla v\|_{S_1}. \quad (2.1)$$

We assume that $T^H(\Omega) = \{\tau_{\Omega}^H\}$ is a regular triangulation of Ω . Here $H = \max_{\tau_{\Omega}^H \in T^H(\Omega)} \{\text{diam}(\tau_{\Omega}^H)\}$ is the mesh size parameter. Let

$$S^H(\Omega) = \{v_H \in C^0(\Omega) : v_H|_{\tau_{\Omega}^H} \in P_{\tau_{\Omega}^H}^r, \forall \tau_{\Omega}^H \in T^H(\Omega)\},$$

be a C^0 -finite element space defined on Ω and $S_0^H(\Omega) = S^H(\Omega) \cap H_0^1(\Omega)$, where $r \geq 1$ is a positive integer and $P_{\tau_{\Omega}^H}^r$ is the space of polynomials of degree not greater than r defined on τ_{Ω}^H . Given $S_1 \subset \Omega$, which aligns with $T^H(\Omega)$, we define $T^H(S_1)$ and $S^H(S_1)$ to be the restriction of $T^H(\Omega)$ and $S^H(\Omega)$ on S_1 .

For these finite element spaces and problem (1.2), we make the following assumptions.

A1 *Interpolant.* There is a finite element interpolation I_H defined on $S^H(\Omega)$ and we denote $\hat{I}_H = I - I_H$ such that for any $w \in H^s(\tau_{\Omega}^H)$, $0 \leq m \leq s \leq r + 1$,

$$\|\hat{I}_H w\|_{m, \tau_{\Omega}^H} \lesssim H^{s-m} |w|_{s, \tau_{\Omega}^H}.$$

A2. Inverse Inequality. For any $w \in S^H(\Omega)$,

$$\|w\|_{1,\Omega} \lesssim H^{-1} \|w\|_{\Omega},$$

A3. Regularity. For any $f \in L^2(\Omega)$, the solutions of

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

satisfies

$$\|u\|_{2,\Omega} \lesssim \|f\|_{L^2(\Omega)}.$$

Now let us state the standard Galerkin equation of (1.2): find $u_H \in S_0^H(\Omega)$ such that

$$a(u_H, v_H) = (f, v_H), \quad \forall v_H \in S_0^H(\Omega). \quad (2.2)$$

And it is classical that

$$\|u - u_H\|_{\Omega} + H \|\nabla(u - u_H)\|_{\Omega} = O(H^{r+1}), \quad (2.3)$$

if $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$, $r \geq 1$.

3 Local and Parallel Two-Grid Scheme

Let us denote

$$\hat{w} = u - u_H \in H_0^1(\Omega).$$

Then the residual equation is

$$a(\hat{w}, v) = (f, v) - a(u_H, v), \quad \forall v \in H_0^1(\Omega). \quad (3.1)$$

Assume that $\{\phi_j\}_{j=1}^N$ is a partition of unity on Ω for given integer $N \geq 1$ such that $\Omega \subset \bigcup_{j=1}^N \text{supp } \phi_j$ and $\sum_{j=1}^N \phi_j \equiv 1$ on Ω . In the rest of the paper, we denote $D_j = \text{supp } \phi_j$ and always assume that D_j aligns with $T^H(\Omega)$. We can rewrite (3.1) as

$$a(\hat{w}, v) = (f, \sum_{j=1}^N \phi_j v) - a(u_H, \sum_{j=1}^N \phi_j v), \quad \forall v \in H_0^1(\Omega).$$

By superposition principle, the above residual equation is equivalent to the summation of the following subproblems:

$$a(\hat{w}^j, v) = (f, \phi_j v) - a(u_H, \phi_j v), \quad \forall v \in H_0^1(\Omega), \quad j = 1, 2, \dots, N. \quad (3.2)$$

That is $\hat{w} = \sum_{j=1}^N \hat{w}^j$. Each subproblem is a "local residual" equation with homogeneous Dirichlet boundary condition, which is driven by right-hand-side term of a very small compact support, and all the subproblems are independent once

u_H is known. To discretize and localize the "local residual" equation and therefore to reduce the computational scale, we restrict the above subproblem in a local domain Ω_j , which contains D_j and is also assumed to be aligned with $T^H(\Omega)$. For each local subdomain Ω_j , we assume that $T^h(\Omega_j) = \{\tau_{\Omega_j}^h\}$ is a regular triangulation on it. Here $h = \max_{1 \leq j \leq N} \max_{\tau_{\Omega_j}^h \in T^h(\Omega_j)} \{\text{diam}(\tau_{\Omega_j}^h)\}$. For simplicity,

the local fine mesh $T^h(\Omega_j)$ is defined as follows throughout the rest of the paper. For a global regular triangulation $T^h(\Omega) = \{\tau_{\Omega}^h\}$ on Ω which aligns with $T^H(\Omega)$, we define $T^h(\Omega_j) = T^h(\Omega)|_{\Omega_j}$. For this mesh parameter h ($h < H$), we introduce following fine mesh finite element spaces $S^h(\Omega_j)$, $S_0^h(\Omega_j)$ and $S^h(\Omega)$, $S_0^h(\Omega)$, which have the same definitions as $S^H(\Omega)$ and $S_0^H(\Omega)$ given in the previous section. Since the functions in $S_0^h(\Omega_j)$ can be extended to functions in $S_0^h(\Omega)$ with zero value outside Ω_j , we regard $S_0^h(\Omega_j)$ as a subspace of $S_0^h(\Omega)$ in the sense of such zero extension. Since Ω_j and $T^h(\Omega)$ align with $T^H(\Omega)$, we always assume

$$S^H(\Omega) \subset S^h(\Omega), \quad S_0^H(\Omega) \subset S_0^h(\Omega) = \bigcup_{1 \leq j \leq N} S_0^h(\Omega_j). \quad (3.3)$$

Now we give the approximate "local residual" equation as follows: find $\hat{w}_{H,h}^j \in S_0^h(\Omega_j)$ such that

$$a(\hat{w}_{H,h}^j, v) = (f, \phi_j v) - a(u_H, \phi_j v), \quad \forall v \in S_0^h(\Omega_j), \quad j = 1, 2, \dots, N. \quad (3.4)$$

It is clear that all subproblems in (3.4) are independent. Note that $\hat{w}_{H,h}^j$ can be extended to the entire domain Ω with zero value outside Ω_j in $H_0^1(\Omega)$, we still use $\hat{w}_{H,h}^j$ to denote such extension in the rest and we denote

$$\hat{w}_{H,h} = \sum_{j=1}^N \hat{w}_{H,h}^j.$$

Now we define the following intermediate approximate solution

$$u_{H,h} = u_H + \hat{w}_{H,h}. \quad (3.5)$$

Since the approximation $u_{H,h}$ is obtained by solving a series of local subproblems which are imposed with artificial homogeneous boundary conditions of the first kind, some local non-physical oscillation may occur. This will certainly bring some bad influence to the global accuracy of the approximation. To diminish such influence, we choose to smooth the above intermediate approximation $u_{H,h}$ by following coarse grid correction: find $E_H \in S_0^H(\Omega)$ such that

$$a(E_H, v) = (f, v) - a(u_{H,h}, v) \quad \forall v \in S_0^H(\Omega). \quad (3.6)$$

And the final approximate solution is defined as

$$u_H^h = u_{H,h} + E_H = u_H + \hat{w}_{H,h} + E_H. \quad (3.7)$$

Now for the implementation of the proposed local and parallel two-grid scheme (3.4)-(3.7), we have to choose a proper partition of unity $\{\phi_j\}_{j=1}^N$ of Ω and its associated computational domain Ω_j . A simple choice of the partition of unity is the piecewise linear Lagrange basis functions associated with the coarse grid triangulation $T^H(\Omega)$, where N is the number of vertices of $T^H(\Omega)$ including the boundary vertices. For each vertex j of the coarse grid, let us denote $D_j = \text{supp } \phi_j$. Then we expand D_j by one coarse mesh layer to get Ω_j , that is

$$\Omega_j = \bigcup_{x_i \in D_j} D_i,$$

where x_i denotes the i th vertex of the coarse mesh triangulation. It is clear that

$$\text{diam}(D_j), \text{dist}(\partial D_j, \partial \Omega_j) \approx H. \quad (3.8)$$

4 Error Estimates

By the idea of fictitious domain method (see [8]), we extend the local sub-problem (3.4) to Ω . Let us denote $\Gamma = \partial\Omega$ and $\Gamma_j = \partial\Omega_j \setminus \Gamma$. If we introduce $H_h^{\frac{1}{2}}(\Gamma_j) = S_0^h(\Omega)|_{\Gamma_j} \subset H^{\frac{1}{2}}(\Gamma_j)$ and $H_h^{-\frac{1}{2}}(\Gamma_j) = (H_h^{\frac{1}{2}}(\Gamma_j))'$ which is equipped with the following norm

$$\|\mu\|_{H_h^{-\frac{1}{2}}(\Gamma_j)} = \sup_{v \in H_h^{\frac{1}{2}}(\Gamma_j)} \frac{\int_{\Gamma_j} v \mu}{\|v\|_{H_h^{\frac{1}{2}}(\Gamma_j)}},$$

we can show that the local residual $\hat{w}_{H,h}^j \subset S_0^h(\Omega)$ satisfies the following saddle point problem: find $(\hat{w}_{H,h}^j, \xi^j) \in S_0^h(\Omega) \times H_h^{-\frac{1}{2}}(\Gamma_j)$ such that $\forall (v, \mu) \in S_0^h(\Omega) \times H_h^{-\frac{1}{2}}(\Gamma_j)$

$$a(\hat{w}_{H,h}^j, v) + \langle \xi^j, v \rangle_j + \langle \mu, \hat{w}_{H,h}^j \rangle_j = (f, \phi_j v) - a(u_H, \phi_j v), \quad (4.1)$$

where

$$\langle \mu, v \rangle_j = \int_{\Gamma_j} \mu v ds \quad \forall \mu \in H_h^{-\frac{1}{2}}(\Gamma_j), v \in S_0^h(\Omega).$$

To show the well-posedness of the above saddle point problem, let us introduce following two finite element spaces

$$\begin{aligned} S_E^h(\Omega_j) &= \{v \in S^h(\Omega_j) : v|_{\partial\Omega_j \setminus \Gamma_j} = 0\}, \\ S_E^h(\Omega \setminus \Omega_j) &= \{v \in S^h(\Omega \setminus \Omega_j) : v|_{\partial(\Omega \setminus \Omega_j) \setminus \Gamma_j} = 0\}. \end{aligned}$$

For given $g \in H_h^{\frac{1}{2}}(\Gamma_j)$, we introduce two auxiliary problems

$$a(u_1, v)_{\Omega_j} = 0, \quad u_1|_{\Gamma_j} = g \quad \forall v \in S_0^h(\Omega_j),$$

and

$$a(u_2, v)_{\Omega \setminus \Omega_j} = 0, \quad u_2|_{\Gamma_j} = g \quad \forall v \in S_0^h(\Omega \setminus \Omega_j).$$

These two problems define two mappings γ_1^{-1} and γ_2^{-1} from $H_h^{\frac{1}{2}}(\Gamma_j)$ into $S_E^h(\Omega_j)$ and $S_E^h(\Omega \setminus \Omega_j)$, respectively. That is

$$u_1 = \gamma_1^{-1}g, \quad u_2 = \gamma_2^{-1}g.$$

And we know that

$$\|\gamma_1^{-1}g\|_{H^1(\Omega_j)}, \|\gamma_2^{-1}g\|_{H^1(\Omega \setminus \Omega_j)} \lesssim \|g\|_{H^{\frac{1}{2}}(\Gamma_j)}.$$

Then we can define an operator γ^{-1} from $H_h^{\frac{1}{2}}(\Gamma_j)$ into $S_0^h(\Omega)$: for any given $g \in H_h^{\frac{1}{2}}(\Gamma_j)$

$$\gamma^{-1}g = \begin{cases} \gamma_1^{-1}g, & \text{in } \Omega_j, \\ \gamma_2^{-1}g, & \text{in } \Omega \setminus \Omega_j. \end{cases}$$

And we have the following property of γ^{-1} :

$$\|\gamma^{-1}g\|_{H_0^1(\Omega)} \lesssim \|g\|_{H^{\frac{1}{2}}(\Gamma_j)} \quad \forall g \in H_h^{\frac{1}{2}}(\Gamma_j).$$

Now for any $\mu \in H_h^{-\frac{1}{2}}(\Gamma_j)$, we have

$$\|\mu\|_{H_h^{-\frac{1}{2}}(\Gamma_j)} = \sup_{g \in H_h^{\frac{1}{2}}(\Gamma_j)} \frac{\langle \mu, g \rangle_j}{\|g\|_{H^{\frac{1}{2}}(\Gamma_j)}} \lesssim \sup_{g \in H_h^{\frac{1}{2}}(\Gamma_j)} \frac{\langle \mu, \gamma^{-1}g \rangle_j}{\|\gamma^{-1}g\|_{H_0^1(\Omega)}} \leq \sup_{v \in S_0^h(\Omega)} \frac{\langle \mu, v \rangle_j}{\|v\|_{H_0^1(\Omega)}}.$$

This ensures that the saddle point problem is well-posed. And it is straight that $\hat{w}_{H,h}^j$ is the solution of this global problem.

Let us recall the residual equation (3.1) and the "local residual" equations (3.2). For the previously defined fine mesh $T^h(\Omega)$ and the associated finite element space $S_0^h(\Omega)$, their fine mesh Galerkin approximations are as follows. Find $\hat{w}_H \in S_0^h(\Omega)$ and $\hat{w}_H^j \in S_0^h(\Omega)$, $j = 1, 2, \dots, N$, such that

$$a(\hat{w}_H, v) = (f, v) - a(u_H, v), \quad \forall v \in S_0^h(\Omega), \quad (4.2)$$

and

$$a(\hat{w}_H^j, v) = (f, \phi_j v) - a(u_H, \phi_j v), \quad \forall v \in S_0^h(\Omega), \quad j = 1, 2, \dots, N. \quad (4.3)$$

And we know that $\hat{w}_H = \sum_{j=1}^N \hat{w}_H^j$ is the Galerkin approximation of \hat{w} in $S_0^h(\Omega)$.

If we denote

$$g_j = \hat{w}_H^j|_{\Gamma_j},$$

we know that \hat{w}_H^j satisfies: $\forall (v, \mu) \in S_0^h(\Omega) \times H_h^{-\frac{1}{2}}(\Gamma_j)$

$$a(\hat{w}_H^j, v) + \langle \zeta^j, v \rangle_j + \langle \mu, \hat{w}_H^j - g_j \rangle_j = (f, \phi_j v) - a(u_H, \phi_j v). \quad (4.4)$$

Here, we can verify the Lagrange multiplier ζ^j satisfies

$$\langle \zeta^j, v \rangle_j = 0, \quad \forall v \in S_0^h(\Omega), \quad (4.5)$$

if we take $\mu = 0$ in (4.4) and compare (4.4) with (4.3).

If we denote by

$$e_{H,h}^j = \hat{w}_H^j - \hat{w}_{H,h}^j, \quad e_{H,h} = \sum_{j=1}^N e_{H,h}^j = u_h - u_{H,h},$$

the "local error" and the global error of $u_{H,h}$ respectively, then by comparing (4.4) with (4.1) and taking $\mu = 0$, we have

$$a(e_{H,h}^j, v) + \langle \xi^j, v \rangle_j = 0, \quad \forall v \in S_0^h(\Omega), \quad j = 1, 2, \dots, N,$$

and

$$a(e_{H,h}, v) + \sum_{j=1}^N \langle \xi^j, v \rangle_j = 0, \quad \forall v \in S_0^h(\Omega). \quad (4.6)$$

It is clear that for all points $x \in \Omega$, there exists a positive integer κ , which has nothing to do with N and x , such that each x belongs to κ different Ω_j at most. By using the previously defined operator γ^{-1} and the fact we just stated, we can easily get the following lemma.

Lemma 4.1: *The multiplier ξ^j in (4.1) satisfies*

$$\|\xi^j\|_{H_h^{-\frac{1}{2}}(\Gamma_j)} \lesssim \|\nabla e_{H,h}^j\|_{\Omega},$$

and

$$\sum_{j=1}^N \langle \xi^j, v \rangle_j \lesssim \kappa^{\frac{1}{2}} \left(\sum_{j=1}^N \|\xi\|_{H_h^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \|v\|_{H^1(\Omega)}.$$

Proof. The first estimate is quite easy if one notices the property of γ^{-1} and we omit its proof. For the second estimate, thanks to the definition of $\|\cdot\|_{H_h^{-\frac{1}{2}}(\Gamma_j)}$, we have $\forall v \in S_0^h(\Omega)$

$$\begin{aligned} \sum_{j=1}^N \langle \xi^j, v \rangle_j &\leq \sum_{j=1}^N \|\xi^j\|_{H_h^{-\frac{1}{2}}(\Gamma_j)} \|v\|_{H^{\frac{1}{2}}(\Gamma_j)} \\ &\lesssim \sum_{j=1}^N \|\xi\|_{H_h^{-\frac{1}{2}}(\Gamma_j)} \|v\|_{H^1(\Omega_j)} \leq \left(\sum_{j=1}^N \|\xi\|_{H_h^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N \|v\|_{H^1(\Omega_j)}^2 \right)^{\frac{1}{2}} \\ &\leq \kappa^{\frac{1}{2}} \left(\sum_{j=1}^N \|\xi\|_{H_h^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \|v\|_{H^1(\Omega)}. \end{aligned}$$

□

Now let us consider the estimation of each $\|\nabla e_{H,h}^j\|_{\Omega}$. To do so, we notice that

$$\|\nabla e_{H,h}^j\|_{\Omega}^2 = \|\nabla(\hat{w}_H^j - \hat{w}_{H,h}^j)\|_{\Omega}^2 = \|\nabla \hat{w}_H^j\|_{\Omega/\Omega_j}^2 + \|\nabla(\hat{w}_H^j - \hat{w}_{H,h}^j)\|_{\Omega_j}^2.$$

It is obvious that $e_{H,h}^j|_{\Omega_j}$ satisfies the following equation

$$a(e_{H,h}^j|_{\Omega_j}, v) = 0, \quad v \in S_0^h(\Omega_j), \quad e_{H,h}^j|_{\partial\Omega_j} = \hat{w}_H^j|_{\partial\Omega_j}.$$

Since $\hat{w}_H^j \in S_0^h(\Omega)$, we have

$$\|e_{H,h}^j\|_{1,\Omega_j} \lesssim \|\hat{w}_H^j\|_{\frac{1}{2},\partial\Omega_j} = \|\hat{w}_H^j\|_{\frac{1}{2},\partial(\Omega \setminus \Omega_j)} \lesssim \|\nabla \hat{w}_H^j\|_{\Omega/\Omega_j}.$$

Then we know that

$$\|\nabla e_{H,h}^j\|_{\Omega}^2 \lesssim \|\nabla \hat{w}_H^j\|_{\Omega/\Omega_j}^2. \quad (4.7)$$

Being aware of (4.7), we give the estimate of $\|\nabla e_{H,h}^j\|_{\Omega}$, which plays a crucial role in this section.

Lemma 4.2: *Let us denote*

$$\alpha_d = \begin{cases} \frac{c}{|\ln H|^2}, & d = 2, \\ \frac{c}{|\ln H|}, & d = 3, \end{cases}$$

where $c > 0$ is a positive constant that does not depend on H , h and Ω_j . Then we have

$$\|\nabla e_{H,h}^j\|_{\Omega}^2 \lesssim H^{2\alpha_d} \|\nabla \hat{w}_H^j\|_{\Omega}^2.$$

Proof. To prove this lemma, we first divide the region $\Omega_j \setminus D_j$ as follows. For example, see Fig. 2.

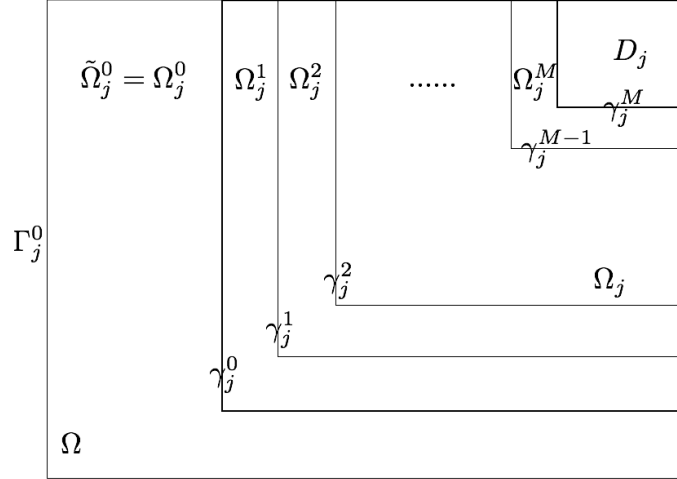


Fig. 2: Division of the region $\Omega_j \setminus D_j$

Let us denote

$$\tilde{\Omega}_j^0 = \Omega_j^0 = \Omega \setminus \Omega_j.$$

We extend the domain $\tilde{\Omega}_j^0$ along the outward normal direction on $\partial\tilde{\Omega}_j^0 \setminus \partial\Omega$ within Ω by a fine mesh layer to obtain $\tilde{\Omega}_j^1 \supset \tilde{\Omega}_j^0$ and denote by Ω_j^1 the incremental annular zone, that is $\Omega_j^1 = \tilde{\Omega}_j^1 \setminus \tilde{\Omega}_j^0$. Repeat the above procedure until we get $\tilde{\Omega}_j^M = \Omega \setminus D_j$, where $M \cong \frac{H}{h}$. Then we obtain a series of subdomains

$$\tilde{\Omega}_j^0 \subset \subset \tilde{\Omega}_j^1 \subset \subset \cdots \subset \subset \tilde{\Omega}_j^M,$$

and a series of disjoint annular zones

$$\Omega_j^0, \Omega_j^1, \dots, \Omega_j^M.$$

It is clear that

$$\tilde{\Omega}_j^k = \bigcup_{i=0}^k \Omega_j^i, \quad k = 0, 1, 2, \dots, M.$$

In what follows, we denote

$$\partial\tilde{\Omega}_j^k = \gamma_j^k \cup \Gamma_j^k, \quad \gamma_j^k = \partial\tilde{\Omega}_j^k / \partial\Omega \quad \text{and} \quad \Gamma_j^k = \partial\tilde{\Omega}_j^k \setminus \gamma_j^k, \quad k = 0, 1, 2, \dots, M.$$

Since

$$a(\hat{w}_H^j, v)_{\tilde{\Omega}_j^k} = 0, \quad \forall v \in S_0^h(\tilde{\Omega}_j^k), \quad k = 1, 2, \dots, M,$$

and $\tilde{\Omega}_j^k = \tilde{\Omega}_j^{k-1} \cup \Omega_j^k$, we know that

$$a(\hat{w}_H^j, v)_{\tilde{\Omega}_j^{k-1}} = -a(\hat{w}_H^j, v)_{\Omega_j^k}, \quad v \in S_0^h(\tilde{\Omega}_j^k). \quad (4.8)$$

We define a smooth function $\psi \in S^h(\tilde{\Omega}_j^k)$ such that $\psi|_{\gamma_j^k} = 0$ for $k \geq 1$ and

$$\text{supp } \psi = \tilde{\Omega}_j^k, \quad \psi(x) \equiv 1 \quad \forall x \in \tilde{\Omega}_j^{k-1}, \quad 0 \leq \psi \leq 1 \quad \text{and} \quad |\nabla \psi(x)| \lesssim h^{-1}.$$

By taking

$$v = I_h(\psi \hat{w}_H^j) \in S_0^h(\tilde{\Omega}_j^k),$$

in (4.8), we derive

$$\begin{aligned} \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}^2 &= a(\hat{w}_H^j, \hat{w}_H^j)_{\tilde{\Omega}_j^{k-1}} = -a(\hat{w}_H^j, I_h(\psi \hat{w}_H^j))_{\Omega_j^k} \\ &\leq \|\nabla \hat{w}_H^j\|_{\Omega_j^k} \|\nabla I_h(\psi \hat{w}_H^j)\|_{\Omega_j^k} \lesssim \|\nabla \hat{w}_H^j\|_{\Omega_j^k} \|\nabla(\psi \hat{w}_H^j)\|_{\Omega_j^k}. \end{aligned} \quad (4.9)$$

For $\|\nabla(\psi \hat{w}_H^j)\|_{\Omega_j^k}$, we have

$$\|\nabla(\psi \hat{w}_H^j)\|_{\Omega_j^k} \leq \|\psi \nabla \hat{w}_H^j\|_{\Omega_j^k} + \|\hat{w}_H^j \nabla \psi\|_{\Omega_j^k} \leq \|\nabla \hat{w}_H^j\|_{\Omega_j^k} + h^{-1} \|\hat{w}_H^j\|_{\Omega_j^k}. \quad (4.10)$$

Now let us estimate $\|\hat{w}_H^j\|_{\Omega_j^k}$. To do so, we introduce the following polar or spherical coordinates (ρ, ω) with origin at j th vertex with respect to 2-D and

3-D case, respectively. Here $\omega = \omega_1$ in 2-D case and $\omega = (\omega_1, \omega_2)$ in 3-D case. The Jacobi determinant of the transformation between the Cartesian coordinate (x_1, \dots, x_d) and (ρ, ω) is

$$J = \frac{D(x_1, \dots, x_d)}{D(\rho, \omega_1, \dots, \omega_{d-1})} = \rho^{d-1} \delta(\omega), \quad \delta(\omega) = \sin^{d-2} \omega_1.$$

In the polar or spherical coordinates, let us denote

$$\Gamma_j^k : \rho(\omega) = \rho_j^k(\omega), \quad \gamma_j^k : \rho(\omega) = \tilde{\rho}_j^k(\omega), \quad 0 \leq k \leq M.$$

It is obvious that

$$\tilde{\rho}_j^k(\omega) \approx H \quad \text{and} \quad H \lesssim \rho_j^k(\omega) \lesssim 1.$$

For $1 \leq k \leq M$ and any point $(\rho, \omega) \in \Omega_j^k$, noting $\hat{w}_H^j \in H_0^1(\Omega)$ and Ω is a convex domain, we have

$$\begin{aligned} |\hat{w}_H^j(\rho, \omega)| &= \left| \int_{\rho_j^k(\omega)}^{\rho} \frac{\partial \hat{w}_H^j}{\partial \rho} d\rho \right| \leq \left(\int_{\rho_j^k}^{\tilde{\rho}_j^k} \frac{1}{\rho^{d-1}} d\rho \right)^{\frac{1}{2}} \left(\int_{\rho_j^k}^{\tilde{\rho}_j^k} \rho^{d-1} \left| \frac{\partial \hat{w}_H^j}{\partial \rho} \right|^2 d\rho \right)^{\frac{1}{2}} \\ &\lesssim \beta_d^{\frac{1}{2}}(H) \left(\int_{\rho_j^k}^{\tilde{\rho}_j^k} \rho^{d-1} \left| \frac{\partial \hat{w}_H^j}{\partial \rho} \right|^2 d\rho \right)^{\frac{1}{2}}, \end{aligned}$$

where $\beta_d(H) = H^{-1}$ when $d = 3$ and $\beta_d(H) = |\ln H|$ when $d = 2$. Thus

$$\begin{aligned} \|\hat{w}_H^j\|_{\Omega_j^k}^2 &= \int_{\Omega_j^k} \rho^{d-1} |\hat{w}_H^j(\rho, \omega)|^2 \delta(\omega) d\rho d\omega \\ &\lesssim \int_{\Omega_j^k} \rho^{d-1} \beta_d(H) \left(\int_{\rho_j^k}^{\tilde{\rho}_j^k} \rho^{d-1} \left| \frac{\partial \hat{w}_H^j}{\partial \rho} \right|^2 d\rho \right) \delta(\omega) d\rho d\omega \\ &\lesssim h H^{d-1} \beta_d(H) \|\nabla \hat{w}_H^j\|_{\Omega_j^k}^2. \end{aligned}$$

Since $\|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^k} \leq \|\nabla \hat{w}_H^j\|_{\Omega_j^k} + \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}$ and $h^{-\frac{1}{2}} H^{\frac{d-1}{2}} \beta_d^{\frac{1}{2}}(H) > 1$, combining the above estimate with (4.9) and (4.10) admits

$$\begin{aligned} \|\hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}^2 &\lesssim \|\nabla \hat{w}_H^j\|_{\Omega_j^k} (\|\nabla \hat{w}_H^j\|_{\Omega_j^k} + h^{-\frac{1}{2}} H^{\frac{d-1}{2}} \beta_d^{\frac{1}{2}}(H) \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^k}) \\ &\leq \|\nabla \hat{w}_H^j\|_{\Omega_j^k}^2 + h^{-\frac{1}{2}} H^{\frac{d-1}{2}} \beta_d^{\frac{1}{2}}(H) \|\nabla \hat{w}_H^j\|_{\Omega_j^k} (\|\nabla \hat{w}_H^j\|_{\Omega_j^k} + \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}) \\ &\lesssim h^{-\frac{1}{2}} H^{\frac{d-1}{2}} \beta_d^{\frac{1}{2}}(H) \|\nabla \hat{w}_H^j\|_{\Omega_j^k}^2 + h^{-\frac{1}{2}} H^{\frac{d-1}{2}} \beta_d^{\frac{1}{2}}(H) \|\nabla \hat{w}_H^j\|_{\Omega_j^k} \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}. \end{aligned}$$

By Young's inequality, we have

$$\|\hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}^2 \lesssim h^{-1} H^{d-1} \beta_d(H) \|\nabla \hat{w}_H^j\|_{\Omega_j^k}^2,$$

or

$$\|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^k}^2 \geq ch H^{1-d} \beta_d^{-1}(H) \|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^{k-1}}^2,$$

where $c > 0$ is a constant that does not depend on H , h , j and k .

By using the last inequality successively, we get

$$\begin{aligned}\|\nabla \hat{w}_H^j\|_{\Omega_j^M}^2 &= \|\nabla \hat{w}_H^j\|_{\Omega_j^{M-1}}^2 + \|\nabla \hat{w}_H^j\|_{\Omega_j^M}^2 \geq (1 + chH^{1-d}\beta_d^{-1}(H))\|\nabla \hat{w}_H^j\|_{\Omega_j^{M-1}}^2 \\ &\geq \cdots \geq (1 + chH^{1-d}\beta_d^{-1}(H))^M \|\nabla \hat{w}_H^j\|_{\Omega_j^0}^2.\end{aligned}$$

Hence

$$\|\nabla \hat{w}_H^j\|_{\Omega_j^0}^2 \leq (1 + chH^{1-d}\beta_d^{-1}(H))^{-M} \|\nabla \hat{w}_H^j\|_{\Omega_j^M}^2.$$

Noting $M \cong \frac{H}{h}$ and $\|\nabla \hat{w}_H^j\|_{\tilde{\Omega}_j^M} \leq \|\nabla \hat{w}_H^j\|_{\Omega}$, simple calculation shows that

$$(1 + chH^{1-d}\beta_d^{-1}(H))^{-M} \cong H^{2\alpha_d}.$$

This, together with (4.7), concludes the proof of the lemma. \square

Following the result in Lemma 4.2, to estimate $\|\nabla e_{H,h}^j\|_{\Omega}^2$, we have to give some estimates of the "local residual" \hat{w}_H^j .

Lemma 4.3: *Suppose the assumptions A1, A2 and A3 are valid. Then for $j = 1, 2, \dots, N$, we have*

$$\|\nabla \hat{w}_H^j\|_{\Omega} \lesssim \|\nabla(u - u_H)\|_{D_j}.$$

Proof. Thanks to the coercive property in (2.1), we can get from (3.2) that

$$\begin{aligned}\|\nabla \hat{w}_H^j\|_{\Omega}^2 &= a(\hat{w}_H^j, \hat{w}_H^j) = (f, \phi_j \hat{w}_H^j) - a(u_H, \phi_j \hat{w}_H^j) \\ &= a(u - u_H, \phi_j \hat{w}_H^j) = a(u - u_H, \hat{I}_H(\phi_j \hat{w}_H^j)) \\ &= a(u - u_H, \hat{I}_H(\phi_j I_H \hat{w}_H^j) + \hat{I}_H[\phi_j \hat{I}_H \hat{w}_H^j]).\end{aligned}$$

From the continuity property of the bilinear form $a(\cdot, \cdot)$, we know

$$\begin{aligned}&a(u - u_H, \hat{I}_H(\phi_j I_H \hat{w}_H^j) + \hat{I}_H[\phi_j \hat{I}_H \hat{w}_H^j]) \\ &\lesssim \|\nabla(u - u_H)\|_{D_j} (\|\nabla[\hat{I}_H(\phi_j I_H \hat{w}_H^j)]\|_{D_j} + \|\nabla \hat{I}_H[\phi_j \hat{I}_H \hat{w}_H^j]\|_{D_j}).\end{aligned}$$

Thanks to **A1**, **A2** and notice that ϕ_j is a linear function on each τ_{Ω}^H and $|D\phi_j| \lesssim H^{-1}$,

$$\begin{aligned}\|\nabla[\hat{I}_H(\phi_j I_H \hat{w}_H^j)]\|_{D_j} &= \left(\sum_{\tau_{\Omega}^H \subset D_j} \|\nabla \hat{I}_H(\phi_j I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2 \right)^{\frac{1}{2}} \\ &\leq H \left(\sum_{\tau_{\Omega}^H \subset D_j} \|D^2(\phi_j I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2 \right)^{\frac{1}{2}} \\ &\lesssim H \left(\sum_{\tau_{\Omega}^H \subset D_j} [\|\phi_j D^2(I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2 + \|D\phi_j D(I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2] \right)^{\frac{1}{2}} \\ &\lesssim H \left(\sum_{\tau_{\Omega}^H \subset D_j} [H^{-2} \|D(I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2 + H^{-2} \|D(I_H \hat{w}_H^j)\|_{\tau_{\Omega}^H}^2] \right)^{\frac{1}{2}} \lesssim \|\nabla \hat{w}_H^j\|_{D_j},\end{aligned}$$

$$\begin{aligned}\|\nabla \hat{I}_H[\phi_j \hat{I}_H \hat{w}_H^j]\|_{D_j} &\lesssim \|D(\phi_j \hat{I}_H \hat{w}_H^j)\|_{D_j} \\ &\lesssim \|D\phi_j \hat{I}_H \hat{w}_H^j\|_{D_j} + \|\phi_j D(\hat{I}_H \hat{w}_H^j)\|_{D_j} \lesssim \|\nabla \hat{w}_H^j\|_{D_j}.\end{aligned}$$

Combination of the above estimates yields

$$\|\nabla \hat{w}_H^j\|_{\Omega} \lesssim \|\nabla(u - u_H)\|_{D_j}.$$

□

Now we give the error estimations of the scheme (3.4)-(3.7) in the following theorem.

Theorem 4.1: *Suppose that assumptions A1, A2, A3 and (2.3) hold and $u \in H^{r+1}(\Omega)$. Then*

$$\|\nabla(u - u_H^h)\|_{\Omega} \lesssim h^r + H^{\alpha_d} \|\nabla(u - u_H)\|_{\Omega},$$

$$\|u - u_H^h\|_{\Omega} \lesssim h^{r+1} + H \|\nabla(u - u_H^h)\|_{\Omega},$$

where $\alpha_d > 0$ is defined in Lemma 4.2.

Proof. First of all, we introduce an H^1 -orthogonal projection P_H from $H_0^1(\Omega)$ onto $S_0^H(\Omega)$: for given $w \in H_0^1(\Omega)$, find $P_H w \in S_0^H(\Omega)$ such that

$$a(v, w - P_H w) = 0, \quad \forall v \in S_0^H(\Omega).$$

It is classical that

$$\|(I - P_H)w\|_{\Omega} \lesssim H \|\nabla w\|_{\Omega}, \quad \forall w \in H_0^1(\Omega).$$

Noticing the definition of $\hat{w}_{H,h}$ and $\sum_{j=1}^N \phi_j = 1$ in Ω , the summation of all the equations of (4.1) with $\mu = 0$ gives the equation satisfied by $\hat{w}_{H,h}$

$$a(\hat{w}_{H,h}, v) = (f, v) - a(u_H, v) - \sum_{j=1}^N \int_{\Gamma_j} \xi^j v ds, \quad \forall v \in S_0^h(\Omega). \quad (4.11)$$

Then we have

$$a(u_{H,h}, v) = (f, v) - \sum_{j=1}^N \int_{\Gamma_j} \xi^j v ds, \quad \forall v \in S_0^h(\Omega).$$

Furthermore, we rewrite the coarse mesh correction as

$$a(E_H, v) = (f, P_H v) - a(u_{H,h}, P_H v), \quad \forall v \in S_0^h(\Omega).$$

Adding the above two equations leads to

$$a(u_H^h, v) = (f, v) + (f, P_H v) - a(u_{H,h}, P_H v) - \sum_{j=1}^N \int_{\Gamma_j} \xi^j v ds, \quad \forall v \in S_0^h(\Omega).$$

Finally, we obtain

$$a(u_H^h, v) = (f, v) - \sum_{j=1}^N \int_{\Gamma_j} \xi^j (I - P_H) v ds, \quad \forall v \in S_0^h(\Omega),$$

and

$$a(u_h - u_H^h, v) = \sum_{j=1}^N \int_{\Gamma_j} \xi^j (I - P_H) v ds, \quad \forall v \in S_0^h(\Omega). \quad (4.12)$$

Thanks to the above error equation of $u_h - u_H^h$, Lemma 4.1, 4.2 and 4.3, we can easily get

$$\|\nabla(u_h - u_H^h)\|_{\Omega} \lesssim H^{\alpha_d} \|\nabla(u - u_H)\|_{\Omega}.$$

Then we can derive the first result by using the triangle inequality.

For the L^2 -error estimate, we use the Aubin-Nitsche duality argument. Since **A3**, for $u_h - u_H^h \in L^2(\Omega)$, there exists $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$a(v, \phi) = (u_h - u_H^h, v), \quad \forall v \in H_0^1(\Omega),$$

and

$$\|\phi\|_{2,\Omega} \lesssim \|u_h - u_H^h\|_{\Omega}.$$

Taking $v = u_h - u_H^h$ and noting (4.12), we have

$$\|u_h - u_H^h\|_{\Omega}^2 = a(u_h - u_H^h, \phi) = a(u_h - u_H^h, (I - P_H)\phi).$$

Thus

$$\begin{aligned} \|u_h - u_H^h\|_{\Omega}^2 &= a(u_h - u_H^h, (I - P_H)\phi) \lesssim \|\nabla(u_h - u_H^h)\|_{\Omega} \|\nabla(I - P_H)\phi\|_{\Omega} \\ &\lesssim H \|\nabla(u_h - u_H^h)\|_{\Omega} \|\phi\|_{2,\Omega} \lesssim H \|\nabla(u_h - u_H^h)\|_{\Omega} \|u_h - u_H^h\|_{\Omega}. \end{aligned}$$

By using triangle inequality, this estimate admits the L^2 -error estimate. \square

From the results in Theorem 4.1, we see that one can improve the convergence order of both H^1 and L^2 errors of the coarse mesh standard Galerkin approximation u_H for α_d order by one two-grid iteration. And it is easy to verify that all the above lemmas and theorem are valid if we replace u_H by u_H^h . This suggests the following two-grid iteration with

$$K = [\alpha_d^{-1} + 0.5] = \begin{cases} O(|\ln H|^2), & d = 2, \\ O(|\ln H|), & d = 3. \end{cases} \quad (4.13)$$

(Step 0) Let $k = 0$ and solve (2.2) to get $u_H \in S_0^H(\Omega)$ and we denote $u_H^{0,h} = u_H$;

(Step 1) Solve the equations in (3.4) with $u_H = u_H^{k,h}$ to get $\{\hat{w}_{H,h}^j\}_{j=1}^N$, which are denoted by $\{\hat{w}_{H,h}^{k+1,j}\}_{j=1}^N$ here. Then we get $u_{H,h}^{k+1}$ by (3.5);

(Step 2) Solve (3.6) with $u_{H,h} = u_{H,h}^{k+1}$ to get E_H^{k+1} and denote

$$u_H^{k+1,h} = u_{H,h}^{k+1} + E_H^{k+1};$$

If $k+1 > K$, stop the iteration and denote $u_H^h = u_H^{k+1,h}$, which is the final approximation with optimal error. Otherwise, let $k := k+1$ and goto (Step 1).

Corollary 4.1: *Suppose $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$, the final approximation u_H^h of the scheme (Step 0)~(Step 2) has the following error bounds*

$$\|\nabla(u - u_H^h)\|_\Omega \lesssim h^r + H^{r+1}, \quad (4.14)$$

$$\|u - u_H^h\|_\Omega \lesssim h^{r+1} + H^{r+2}. \quad (4.15)$$

It is obvious that, to get the optimal H^1 or L^2 error, we should configure H and h such that

$$h \sim H^{\frac{r+1}{r}} \quad \text{or} \quad h \sim H^{\frac{r+2}{r+1}}, \quad (4.16)$$

respectively.

5 Numerical Experiments

In this section, we give some numerical examples to verify the analysis results. For simplicity, in all numerical examples we consider the following piecewise linear finite element spaces, that is $r = 1$:

$$S^H(\Omega) = \{v \in C^0(\Omega) : v|_{\tau_\Omega^H} \in P_{\tau_\Omega^H}^1, \forall \tau_\Omega^H \in T^H(\Omega)\}, \quad S_0^H(\Omega) = S^H(\Omega) \cap H_0^1(\Omega).$$

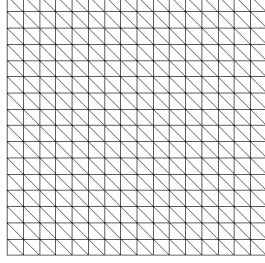
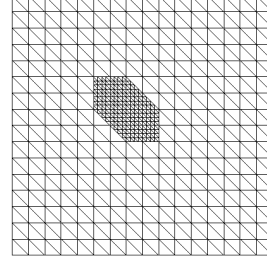
According to (4.16), to reach the H^1 accuracy of the standard Galerkin approximation in $S_0^h(\Omega)$, we choose H and h such that $h \sim H^2$. In this case

$$\|\nabla(u - u_H^h)\|_\Omega = O(H^2). \quad (5.1)$$

On the other hand, to reach the L^2 accuracy of the standard Galerkin approximation in $S_0^h(\Omega)$, we choose H and h such that $h \sim H^{\frac{3}{2}}$. With such configuration, we have

$$\|u - u_H^h\|_\Omega = O(H^3). \quad (5.2)$$

In the following, we try to verify (5.1)-(5.2) and the efficiency of the proposed iteration scheme (Step 0)-(Step 2) in last section by some numerical experiments. On the one hand, since all the subproblems in (Step 1) are independent with each other once the coarse mesh approximation u_H is closed at hand, there will be no communication cost when solving them simultaneously in parallel computer systems. The only thing one should pay attention is the computing of the right hand side in (Step 1) and (Step 2). In (Step 1) we only need the coarse mesh information on D_j , the support of ϕ_j . And in (Step 2), for calculating the second term of the right hand side of (3.6), we calculate it in each fine mesh

Fig. 3: 2-D coarse mesh $T^H(\Omega)$ Fig. 4: 2-D local fine mesh $T^h(\Omega_j)$

element, which dramatically increases the computing time compared with the standard coarse mesh Galerkin method. How to calculate this term efficiently is somehow critical to make the scheme more efficient.

On the other hand, very large parallel computer systems that have huge amount of computing cores give us a possibility to deal with large scale computation and get very accurate approximation. In this case, the scale of the coarse mesh standard Galerkin scheme and the coarse mesh correction problem could be very large and how to solve the coarse mesh problem could be the bottleneck of the entire iterative scheme. Therefore, we should use some parallel solver to cope with the correction step, for example the algebraic multi-grid method which is well known by its efficiency. In the following numerical experiment, we will compare the numerical performance of the proposed iterative scheme in last section with the fine grid standard Galerkin scheme. For the fine grid standard Galerkin scheme, we use pARMS (parallel Algebraic Multilevel Solver) as the parallel sparse solver. In the following numerical experiments, the scale of both the fine mesh subproblems and the coarse mesh correction problem are not very large, so we only use direct method for solving them, although some parallel sparse solvers could be applied to the coarse mesh correction step when the scale of the coarse mesh correction is large. And it is shown that the efficiency of the proposed iteration scheme is higher than the fine mesh standard Galerkin scheme with the previously mentioned parallel sparse solver.

First, we consider two 2-D examples. In these two examples, the domain is the unit square $\Omega = (0, 1) \times (0, 1)$ with a uniform triangulation $T^H(\Omega) = \{\tau_\Omega^H\}$, see Fig. 3. The Fig. 4 is the local fine mesh defined on the j th coarse mesh node.

In the first 2-D example, we consider the problem with the following analytic solution

$$u(x, y) = 100(x^2 - 2x^3 + x^4)(y - 3y^2 + 2y^3).$$

In this case, we can get the exact error of the numerical solution.

In the following Tab. 1 and Tab. 3, we give some numerical results according

to the above configurations of H and h . And in Tab. 2 and Tab. 4 we give the CPU time comparison for the fine mesh standard Galerkin method and the scheme proposed in this paper with respect to getting optimal H^1 and L^2 errors, respectively. That is in Tab. 2 we show the CPU time used when $h = H^2$ and in Tab. 4 we show the CPU time used when $h = H^{\frac{3}{2}}$. For numerical experiments that the true solution u is known, we define the convergence order "ORDER₁" with respect to the coarse mesh size H as

$$\text{ORDER}_1(u_{app}) = \begin{cases} 1 + \frac{\ln \frac{\|\nabla(u - u_H)\|_{0,\Omega}}{\|\nabla(u - u_{app})\|_{0,\Omega}}}{|\ln H|}, & H^1 \text{ error order,} \\ 2 + \frac{\ln \frac{\|u - u_H\|_{0,\Omega}}{\|u - u_{app}\|_{0,\Omega}}}{|\ln H|}, & L^2 \text{ error order.} \end{cases}$$

The symbol u_{app} stands for certain approximation of u defined in the algorithm. And "Iteration" stands the number of iterations that are taken for obtaining the final approximation.

Tab. 1: H^1 -error ($h = H^2$)

$\frac{1}{H}$	8	16	32
$\ \nabla(u - u_H)\ _{0,\Omega}$	6.8371×10^{-1}	3.4949×10^{-1}	1.7574×10^{-1}
$\ \nabla(u - u_h)\ _{0,\Omega}$	8.7995×10^{-2}	2.2009×10^{-2}	5.5023×10^{-3}
$\ \nabla(u - u_{H,h}^1)\ _{0,\Omega}$	1.6292×10^{-1}	6.7335×10^{-2}	2.3873×10^{-2}
$\ \nabla(u - u_H^h)\ _{0,\Omega}$	8.8087×10^{-2}	2.2041×10^{-2}	5.5291×10^{-3}
ORDER ₁ ($u_{H,h}^1$)	1.69	1.59	1.58
ORDER ₁ (u_H^h)	1.98	1.9968	1.9981
Iteration	2	2	2

Tab. 2: CPU time comparison ($h = H^2$, $H^{-1} = 32$)

NP	4	8	16
CPU time (u_h)	1532.71s	940.56s	559.12s
CPU time (u_H^h)	531.67s	265.92s	133.05

The second 2-D example is defined by giving

$$f = 70 \log((x + 0.1)(\sin \pi y + 1)).$$

In this example, since the exact solution u is unknown, the convergence order of the approximate solution is calculated as

$$\text{ORDER}_2(u_{app}) = \begin{cases} \min\{2, 1 + \frac{\ln \frac{\|\nabla(u_h - u_H)\|_{0,\Omega}}{\|\nabla(u_h - u_{app})\|_{0,\Omega}}}{|\ln H|}\}, & H^1 \text{ error order,} \\ \min\{3, 2 + \frac{\ln \frac{\|u_h - u_H\|_{0,\Omega}}{\|u_h - u_{app}\|_{0,\Omega}}}{|\ln H|}\}, & L^2 \text{ error order.} \end{cases}$$

Tab. 3: L^2 -error ($h = H^{\frac{3}{2}}$)

$\frac{1}{H}$	25	36	49	64
$\ u - u_H\ _{0,\Omega}$	3.3784×10^{-3}	1.6347×10^{-3}	8.8361×10^{-4}	5.1832×10^{-4}
$\ u - u_h\ _{0,\Omega}$	1.3597×10^{-4}	4.5545×10^{-5}	1.8145×10^{-5}	8.1066×10^{-6}
$\ u - u_H^h\ _{0,\Omega}$	1.1498×10^{-4}	3.8173×10^{-5}	1.5086×10^{-5}	7.5306×10^{-6}
ORDER ₁ (u_H^h)	3.05	3.05	3.05	3.02
Iteration	1	1	2	2

Tab. 4: CPU time comparison ($h = H^{\frac{3}{2}}$, $H^{-1} = 64$)

NP	4	8	16
CPU time (u_h)	86.56s	63.17s	49.47s
CPU time (u_H^h)	119.34s	64.76s	36.12s

Here u_h is the standard Galerkin approximation in the fine mesh finite element space $S_0^h(\Omega)$ and the symbol u_{app} stands for certain approximation of u defined in the algorithm. Since, for example, the H^1 error estimate of the fine mesh standard Galerkin approximation admits the following estimation when $h = H^2$

$$\|\nabla(u - u_h)\|_{0,\Omega} = O(h) = O(H^2),$$

the “ORDER₂(u_{app})” calculated by the above formula equals to 2 means

$$\|\nabla(u_h - u_{app})\|_{0,\Omega} = O(H^2),$$

therefore

$$\|\nabla(u - u_{app})\|_{0,\Omega} = O(H^2).$$

The Tab. 5 and Tab. 6 give the numerical results of this test problem.

Tab. 5: H^1 -error ($h = H^2$)

$\frac{1}{H}$	8	16	32
$\ \nabla(u_h - u_H)\ _{0,\Omega}$	1.8407×10^0	9.7353×10^{-1}	4.9548×10^{-1}
$\ \nabla(u_h - u_{H,h}^1)\ _{0,\Omega}$	3.1398×10^{-1}	1.6695×10^{-1}	7.3657×10^{-2}
$\ \nabla(u_h - u_H^h)\ _{0,\Omega}$	9.4723×10^{-2}	4.1034×10^{-2}	9.4765×10^{-4}
ORDER ₂ ($u_{H,h}^1$)	1.85	1.64	1.55
ORDER ₂ (u_H^h)	2	2	2
Iteration	1	1	2

In the rest of this section, we will give two 3-D numerical examples. In these two examples, the domain Ω is the unit cube $(0,1)^3$. The Fig. 5 and 6 give some free sketches of the 3-D coarse mesh and associated local fine mesh.

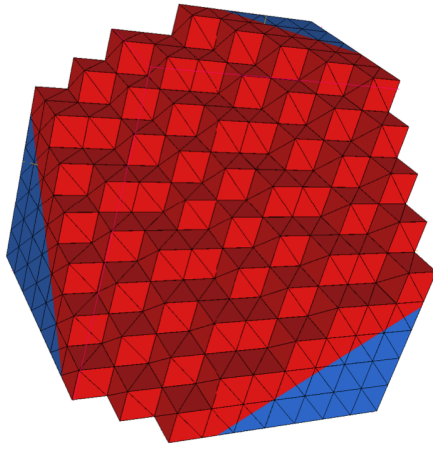
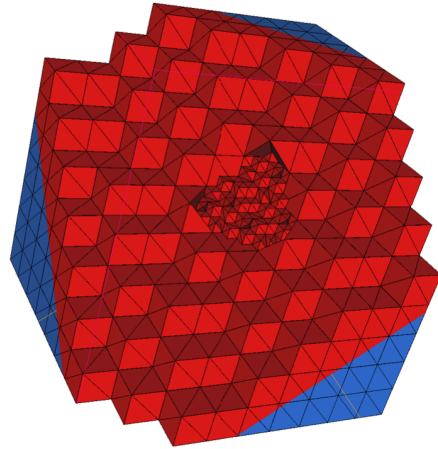
Tab. 6: L^2 -error ($h = H^{\frac{3}{2}}$)

$\frac{1}{H}$	25	36	49	64
$\ u_h - u_H\ _{0,\Omega}$	8.0991×10^{-3}	3.9813×10^{-3}	2.1717×10^{-3}	1.2812×10^{-4}
$\ u_h - u_H^h\ _{0,\Omega}$	2.6319×10^{-4}	4.5438×10^{-6}	1.9168×10^{-6}	1.1412×10^{-6}
$\text{ORDER}_2(u_H^h)$	3	3	3	3
Iteration	1	2	2	2

The first 3-D example is a test problem with the following analytic solution

$$u(x, y, z) = 100(x^2 - 2x^3 + x^4)(y - 3y^2 + 2y^3)(z^3 - z).$$

The numerical results are given in Tab.7 and 8.

Fig. 5: 3-D coarse mesh $T^H(\Omega)$ Fig. 6: 3-D local fine mesh $T^h(\Omega_j)$

The second example of 3-D case is a test problem driven by the following free term

$$f = 70 \log((x + 0.1)(\sin \pi y + 1)(z + 0.1)(\sin \pi z + 1)),$$

whose numerical results are given in Tab.9 and 10.

All the above numerical results are obtained by using the public domain software FreeFem++ [24].

Tab. 7: H^1 -error ($h = H^2$)

$\frac{1}{H}$	6	8	10	12
$\ \nabla(u - u_H)\ _{0,\Omega}$	2.9515×10^{-1}	2.2774×10^{-1}	1.8467×10^{-1}	1.5504×10^{-1}
$\ \nabla(u - u_h)\ _{0,\Omega}$	5.9048×10^{-2}	3.3496×10^{-2}	2.1515×10^{-2}	1.4969×10^{-2}
$\ \nabla(u - u_{H,h}^1)\ _{0,\Omega}$	8.5886×10^{-2}	6.2379×10^{-2}	4.7159×10^{-2}	3.6729×10^{-2}
$\ \nabla(u - u_H^h)\ _{0,\Omega}$	5.9436×10^{-2}	3.3868×10^{-2}	2.1682×10^{-2}	1.5110×10^{-2}
$\text{ORDER}_1(u_{H,h}^1)$	1.69	1.62	1.59	1.58
$\text{ORDER}_1(u_H^h)$	1.89	1.92	1.93	1.94
Iteration	2	2	2	2

Tab. 8: L^2 -error ($h = H^{\frac{3}{2}}$)

$\frac{1}{H}$	9	16	25
$\ u - u_H\ _{0,\Omega}$	8.6338×10^{-3}	2.8568×10^{-3}	1.1850×10^{-3}
$\ u - u_h\ _{0,\Omega}$	1.2840×10^{-3}	2.3929×10^{-4}	6.3981×10^{-5}
$\ u - u_H^h\ _{0,\Omega}$	1.3024×10^{-3}	2.3795×10^{-4}	6.2568×10^{-5}
$\text{ORDER}_1(u_H^h)$	2.86	2.90	2.91
Iteration	2	2	2

Remark From the construction of the partition of unity used in the algorithm, we see that the computational domain of each local subproblem is contained in a ball with radius of $O(H)$. This means that the volume of the computational domain of each local subproblem tends to zero as the coarse mesh size H tends to zero. In this sense, we call the algorithm given in this paper an expandable local and parallel two-grid finite element algorithm.

Acknowledgment

This work is supported by NSFC (Grant No. 11571274 & 11171269) and the Ph.D. Programs Foundation of Ministry of Education of China (Grant No. 20110201110027). And we would like to thank the anonymous referees for their valuable suggestions and comments.

References

- [1] R.A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] A. AIT OU AMMI AND M. MARION, *Nonlinear Galerkin Methods and Mixed Finite Elements: Two-Grid Algorithms for the Navier-Stokes Equations*, Numer. Math., 68(1994), 189–213.

Tab. 9: H^1 -error ($h = H^2$)

$\frac{1}{H}$	6	8	10	12
$\ \nabla(u_h - u_H)\ _{0,\Omega}$	3.1859×10^0	2.5719×10^0	2.1384×10^0	1.8236×10^0
$\ \nabla(u_h - u_{H,h}^1)\ _{0,\Omega}$	5.9544×10^{-1}	5.1948×10^{-1}	4.3693×10^{-1}	3.6855×10^{-1}
$\ \nabla(u_h - u_H^h)\ _{0,\Omega}$	5.5748×10^{-2}	4.0252×10^{-2}	2.9719×10^{-2}	2.2962×10^{-2}
$\text{ORDER}_2(u_{H,h}^1)$	1.94	1.77	1.69	1.64
$\text{ORDER}_2(u_H^h)$	2	2	2	2
Iteration	2	2	2	2

Tab. 10: L^2 -error ($h = H^{\frac{3}{2}}$)

$\frac{1}{H}$	9	16	25
$\ u_h - u_H\ _{0,\Omega}$	7.7114×10^{-2}	2.8568×10^{-2}	1.2454×10^{-2}
$\ u_h - u_H^h\ _{0,\Omega}$	5.7182×10^{-3}	1.5811×10^{-3}	4.2787×10^{-5}
$\text{ORDER}_2(u_H^h)$	3	3	3
Iteration	1	1	2

- [3] I. BABUSKA AND J.M. MELENK, *The Partition of Unity Method*, International Journal for Numerical Methods in Engineering, 40(1997), 727758.
- [4] R.E. BANK, *Hierarchical Bases and the Finite Element Method*, Acta Numerica, 5(1996), 1–43.
- [5] T. CHAN AND T. MATHEW, *Domain Decomposition Algorithms*, Acta Numerica, 3(1994), 61–143.
- [6] P.G. CIARLET AND J.L. LIONS, *Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part I)*, North-Holland, Amsterdam, 1991.
- [7] C. FOIAS, O. MANLEY AND R. TEMAM, *Modeling of the Interaction of Small and Large Eddies in Two-Dimensional Turbulent Flow*, Math. Mod. Numer. Anal., 22(1988), 93–114.
- [8] V. GIRAULT, R. GLOWINSKI, H. LÓPEZ AND J.-P. VILA, *A boundary multiplier/fictitious domain method for the steady incompressible Navier-Stokes equations*, Numer. Math., 88(2001), 75–103.
- [9] W. HACKBUSCH, *Multi-grid Methods and Applications*, Springer, New York, 1985.
- [10] Y. HE, J. XU AND A. ZHOU, *Local and parallel finite element algorithms for the Navier-Stokes problem*, Journal of Computational Mathematics, 24(2006), 227–238.

- [11] Y. HE, J. XU AND A. ZHOU, *Local and parallel finite element algorithms for the Stokes problem*, Numer. Math. 109:3(2008), 415–434.
- [12] Y. HOU AND K. LI, *Tangent Space Correction Method for the Galerkin Approximation Based on Two-grid Finite Element*, Applied Mathematics and Computation, 175:1(2006), 413–429.
- [13] Y. HOU AND K. LI, *Post-processing Fourier Galerkin Method for the Navier-Stokes Equations*, SIAM J. Numer. Anal., 47:3(2009), 1909–1922.
- [14] M.G. LARSON AND A. MÅLQVIST, *Adaptive Variational Multi-scale Methods Based on a Posteriori Error Estimation: Energy Norm Estimates for Elliptic Problems*, Computer Methods in Applied Mechanics and Engineering, 196(2007), 2313–2324.
- [15] M.G. LARSON AND A. MÅLQVIST, *An Adaptive Variational Multi-scale Method for Convection-Diffusion Problems*, Communications in Numerical Methods in Engineering, 25(2009), 65–79.
- [16] K. LI AND Y. HOU, *An AIM and One-step Newton Method for the Navier-Stokes Equations*, Computer Methods in Applied Mechanics and Engineering, 190(2001), 6141–6155.
- [17] Q. LIU AND Y. HOU, *A POST-PROCESSING MIXED FINITE ELEMENT METHOD FOR NAVIER-STOKES EQUATIONS*, International Journal of Computational Fluid Dynamics, 23:6(2009), 461–475.
- [18] J. NITSCHKE AND A.H. SCHATZ, *Interior Estimates for Ritz-Galerkin Methods*, Math. Comp., 28(1974), 937–955.
- [19] C. WANG, Z. HUANG AND L. LI, *Two-Grid Partition of Unity Method for Second Order Elliptic Problems*, Applied Mathematics and Mechanics, 29(2008), 527–533.
- [20] J. XU, *Iterative Methods by Space Decomposition and Subspace Correction*, SIAM Review, 34(1992), 581–613.
- [21] J. XU AND A. ZHOU, *Local and parallel finite element algorithms based on two-grid discretizations*, Math. Comput., 69(2000), 881–909.
- [22] J. XU AND A. ZHOU, *Local and Parallel Finite Element Algorithms Based on Two-Grid Discretizations for Nonlinear Problems*, Advances in Computational Mathematics, 14(2001), 293–327.
- [23] H. ZHENG, L. SONG, Y. HOU AND Y. ZHANG, *The Partition of Unity Parallel Finite Element Algorithm*, Advances in Computational Mathematics, DOI 10.1007/s10444-014-9392-x.
- [24] FreeFem++, Ver. 3.17. Available at: <http://www.freefem.org/ff++/>.